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LETTER TO THE EDITOR

Integrable non-isospectral flows associated with the Kadomtsev–Petviashvili equations in 2+1 dimensions

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Abstract. The symmetries of the Kadomtsev-Petviashvili (KP) equations in 2+1 dimensions yield *two* hierarchies of integrable non-linear evolution equations (NEE): one is the familiar family of isospectral flows—the KP hierarchy. The other is non-isospectral and its flows have coefficients which depend linearly on x and y. The spectral methods used to solve KP can be used to solve all these NEE. An underlying infinite-dimensional Lie algebra is used to determine all the Lax pairs for both families, and it also determines their symplectic structures. Constants of the motion are constructed for the non-isospectral cases.

We take the two KP equations [1-3] in the forms [2-7]‡:

$$u_t = K[u] = \pm D_x^{-1} u_{yy} - 6uu_x - u_{xxx} \tag{1}$$

$u_x \equiv \partial u/\partial x$, etc; $D_x \equiv \partial/\partial x$ and $D_x^{-1} \equiv \int_{-\infty}^{x} dx'$; KP-I is the positive sign in (1), KP-II is the negative sign. If $u \to 0$ sufficiently rapdily as $x^2+y^2 \to \infty$ and $\int_{-\infty}^{\infty} u\, dx = 0$, KP-I is solved by a non-local Riemann-Hilbert problem method [4-6]; KP-II is solved by a $\bar{\partial}$ method [5, 6]. Both equations are Hamiltonian and are completely integrable in the sense of Liouville-Arnold [7]. There are infinite sets of independent commuting constants [2, 3, 7, 8] and Noether's theorem then suggests there are infinite sets of symmetries leaving the Lagrangian invariant. Two such sets, which we call K- and τ-symmetries, are known [9, 10]. The first few are [10]:

$$K_0 = \tfrac{1}{3}u_x \qquad K_1 = \tfrac{2}{3}u_y \qquad K_2 = K \equiv D_x^{-1}u_{yy} - 6uu_x - u_{xxx}$$

$$K_3 = -4u_{yxx} + \tfrac{4}{3}(D_x^{-1}u_{yyy}) - 8u_x D_x^{-1}u_y + 6uu_y \tag{2}$$

$$\tau_0 = tK_0 + \sigma_0 \qquad \tau_1 = tK_1 + \sigma_1 \qquad \tau_2 = tK_2 + \sigma_2 \tag{3a}$$

with

$$\sigma_1 = yK_0 \qquad \sigma_2 = xK_0 + yK_1 + \tfrac{2}{3}u \qquad \sigma_3 = xK_1 + yK_2 + \tfrac{4}{3}D_x^{-1}u_y$$

$$\sigma_4 = xK_2 + yK_3 - 8u^2 - 4u_{xx} - 2u_x D_x^{-1}u + 2D_x^{-2}u_{yy}. \tag{3b}$$

Since the K- and τ-symmetries are symmetries leaving the KP-I equations (1) invariant, they must obviously satisfy the linear variation equation (given here for the τ)

$$\tau_t = K'(\tau) \equiv \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} K[u + \varepsilon\tau]. \tag{4}$$

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‡ It is here convenient to take the scaling of KP as in [8] rather than as in, e.g., [10].

$K'(\tau)$ is thus the Gateaux derivative of K, the functional $K[u]$ of u in (1), in the direction of τ. From the Lie bracket defined as

$$[\![A, B]\!] = A'(B) - B'(A) \tag{5}$$

for any two functionals $A[u]$, $B[u]$ of u, the higher-order K- and τ-symmetries are obtained [10] through

$$K_{n+1} = \frac{3}{n+1}[\![K_n, \tau_3]\!] \qquad \tau_{n+1} = \frac{3}{n-3}[\![\tau_n, \tau_3]\!] \tag{6}$$

with K_3 from (2) and $n \geqslant 3$, and τ_4 from (3) with $n \geqslant 4$. More generally they form the infinite-dimensional Lie algebra [10]

$$\begin{aligned} &[\![K_m, K_n]\!] = 0 \qquad [\![K_m, \tau_n]\!] = \tfrac{1}{3}(m+1)K_{m+n-2} \\ &[\![\tau_m, \tau_n]\!] = \tfrac{1}{3}(m-n)\tau_{m+n-2} \qquad m, n = 0, 1, 2, \ldots \end{aligned} \tag{7}$$

so the σ_n (which are *not* symmetries) satisfy the same algebra with σ_n replacing τ_n. The algebras of the σ_n or τ_n alone are of Virasoro type [11].

It is proved [12] that there exist I_n, J_n such that $K_n = \mathrm{D}_x \delta I_n / \delta u$, $\sigma_n = \mathrm{D}_x \delta J_n / \delta u$ where $\delta/\delta u$ is the functional derivative. The first few of these are

$$\begin{aligned} &I_0 = \tfrac{1}{2}\int u^2 \,\mathrm{d}x\,\mathrm{d}y \qquad I_1 = \tfrac{1}{3}\int (u\mathrm{D}_x^{-1}\mathrm{D}_y u)\,\mathrm{d}x\,\mathrm{d}y \\ &I_2 = \tfrac{1}{2}\int [(\mathrm{D}_x^{-1}\,\mathrm{D}_y u)^2 + u_x^2 - 2u^3]\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{8}$$

and

$$\begin{aligned} &J_0 = -\tfrac{1}{18}\int xu\,\mathrm{d}x\,\mathrm{d}y \qquad J_1 = \tfrac{1}{6}\int yu^2\,\mathrm{d}x\,\mathrm{d}y \\ &J_2 = \int (\tfrac{1}{6}xu^2 + \tfrac{1}{3}uy\mathrm{D}_x^{-1}\mathrm{D}_y u)\,\mathrm{d}x\,\mathrm{d}y \\ &J_3 = \int [\tfrac{1}{2}y(\mathrm{D}_x^{-1}\mathrm{D}_y u)^2 + \tfrac{1}{2}u_x^2 - u^3 + \tfrac{1}{3}xu\mathrm{D}_x^{-1}\mathrm{D}_y u]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{9}$$

They form an infinite-dimensional Lie algebra isomorphic to (7) through the correspondence [12] $K_n \leftrightarrow I_n$, $\tau_n \leftrightarrow J_n$ and bracket the Poisson bracket. More precisely this arises through the correspondence (the equality) $[\![A, B]\!] = \mathrm{D}_x(\delta\{F, G\}/\delta u)$ with $A = \mathrm{D}_x(\delta F/\delta u)$, etc: in both cases the bracket is the usual Poisson bracket [7, 12] $\{F, G\} = \int [(\delta F/\delta u)\mathrm{D}_x(\delta G/\delta u)]\,\mathrm{d}x$ for KP. Since I_2 is the Hamiltonian of KP-I [7, 12], and the I_n are (plainly) independent and in involution, $u_t = \mathrm{D}_x(\delta I_n/\delta u) = K_n$, $n = 2, 3, \ldots$, form the well known hierarchy of integrable KP-I equations. For the J_n one can check that when $u_t = K_2$ the *total* derivatives $\mathrm{d}(tI_n + J_n)/\mathrm{d}t = 0$ and $tI_n + J_n$ are constants. Likewise if $u_t = K_m$, then $\tfrac{1}{3}(m+1)tI_{m+n-2} + J_n$ are constants. Even so, one asks whether the equations $u_t = \mathrm{D}_x(\delta J_n/\delta u) = \sigma_n$, $n = 2, 3, \ldots$, have significance. The point of this letter is to show that these flows too are integrable flows and can be integrated.

We note that there are also two sorts of symmetries, K_n and τ_n, for the AKNS systems in 1+1 dimensions, and Li and Zhu [13] show that $u_t = K_n$ and $u_t = \sigma_n$ are isospectral and non-isospectral respectively. The non-isospectral flows have coefficients depending explicitly, but linearly, on x—'explaining' previous work [14, 15 and references therein]† on this feature. The AKNS systems are a reduction of the KP-I system [17] and apparently KP is more fundamental. In this letter we show the flows $u_t = K_n$ (K_n from (2)) are the usual isospectral KP-I hierarchy while flows $u_t = \sigma_{N+2}$, $N = n-2$, are non-isospectral and are hierarchies of generalised KP-I equations with coefficients depending linearly on both x and y. It is convenient to introduce the label $N = n-2$ (see below) though $u_t = \sigma_{N+2}$ means $u_t = \sigma_n$.

The flows $u_t = \sigma_{N+2}$ with coefficients depending on x and y are integrable by the same non-local Riemann–Hilbert problem as integrates all of the KP-I hierarchy $u_t = K_{N+2}$. Moreover, unlike the situation for KdV in 1+1 dimensions, and illustrating the more fundamental nature of the KP system, the flows $u_t = \sigma_{N+2}$ are Hamiltonian with infinite sets of commuting constants $\bar{I}_m^{(N)}$, different for each N, which we give explicitly; there are further quantities $\bar{J}_m^{(N)}$, and there are symmetries $\bar{K}_m^{(N)}$, $\bar{\tau}_m^{(N)}$ such that $\bar{K}_m^{(N)} = \mathrm{D}_x \delta \bar{I}_m^{(N)}/\delta u$, $\bar{\tau}_m^{(N)} = \mathrm{D}_x \delta \bar{J}_m^{(N)}/\delta u$. These form a Lie algebra (for fixed N) [18]:

$$\begin{aligned} &[\![\bar{K}_m^{(N)}, \bar{K}_l^{(N)}]\!] = 0 \qquad &&[\![\bar{K}_m^{(N)}, \bar{\tau}_l^{(N)}]\!] = \tfrac{1}{3}(m+1)\bar{K}_{m+l+N}^{(N)} \\ &[\![\bar{\tau}_m^{(N)}, \bar{\tau}_l^{(N)}]\!] = \tfrac{1}{3}(m-l)\bar{K}_{m+l+N}^{(N)} \qquad &&m, l = 0, 1, 2, \dots . \end{aligned} \tag{10}$$

We show in this letter how the eigenspectrum and spectral data for $u_t = \sigma_{N+2}$ evolve in time. We also show how the Lie algebras (7) can be used to derive Lax pairs for the flow $u_t = K_{N+2}$ *and* $u_t = \sigma_{N+2}$. This is important because the usual integrodifferential operator (familiar as L^+ [2] in 1+1 dimensions) does not always exist in 2+1 dimensions in conventional form [19–21]. We give actual examples of some integrable non-isospectral flows in the final part of this letter. We focus on KP-I throughout but note that there is a similar analysis for KP-II although a $\bar{\partial}$ problem is needed to solve these equations.

We will now consider the Lax pairs. The KP-I equation (1) is the compatibility condition $L_t = [B, L]$ on $L\psi = 0$, $\psi_t = B\psi$ with the Lax pair [2–7]

$$L = \tfrac{1}{3}\mathrm{i}\sqrt{3}\,\mathrm{D}_y + \mathrm{D}_x^2 + u \tag{11a}$$

$$B = -4\mathrm{D}_x^3 - 6u\mathrm{D}_x - 3u_x + \mathrm{i}\sqrt{3}\,\mathrm{D}_x^{-1}u_x \tag{11b}$$

where [,] is the usual commutator. One checks that $L_t = [B_n, L]$, $n = 0, 1, 2, 3$ and $L_t = [C_n, L]$, $n = 1, 2, 3, 4$ are the non-linear evolution equations (NEE) $u_t = K_n[u]$ and $u_t = \sigma_n[u]$ respectively when

$$B_0 = \tfrac{1}{3}\mathrm{D}_x \qquad B_1 = \mathrm{i}\tfrac{1}{3}2\sqrt{3}(\mathrm{D}_x^2 + u) \qquad B_2 = B \text{ (of (11}b\text{))}$$

$$\begin{aligned} B_3 = &-\mathrm{i}8\sqrt{3}\,\mathrm{D}_x^4 - \mathrm{i}16\sqrt{3}u\,\mathrm{D}_x^2 - (\mathrm{i}16\sqrt{3}\,u_x + 8\mathrm{D}_x^{-1}u_y)\mathrm{D}_x \\ &-(\mathrm{i}8\sqrt{3}\,u^2 + \mathrm{i}8\sqrt{3}\,u_{xx} + 4u_y - \tfrac{4}{3}\sqrt{3}\,\mathrm{i}D_x^{-2}u_{yy}) \end{aligned} \tag{12}$$

$$C_1 = yB_0 - (\mathrm{i}/6\sqrt{3})x \qquad C_2 = yB_1 + xB_0 + \tfrac{1}{3}$$

$$C_3 = yB_2 + xB_1 + \tfrac{1}{3}2\mathrm{i}\sqrt{3}\,\mathrm{D}_x + \tfrac{1}{3}\mathrm{i}\sqrt{3}\,\mathrm{D}_x^{-1}u$$

$$C_4 = yB_3 + xB_2 - 6\mathrm{D}_x^2 + \tfrac{2}{3}\mathrm{i}\sqrt{3}\mathrm{D}_x^{-1}u_y 4u.$$

These operator polynomials are found directly by equating coefficients of powers D_x (note that $\mathrm{D}_x^{-1}u_y$ is a scalar number not an operator). To proceed further this way

† An error in [15] is corrected in [16].

would be impossibly tedious. However, we can find all of the B_n and C_n from the algebra (7). We define operators B_{n+1}, polynomials in D_x, through

$$B_{n+1}=\frac{3}{n+1}(B'_n(\sigma_3)-C'_3(K_n)+[B_n, C_3]) \qquad n\geqslant 3 \tag{13}$$

where σ_3 and C_3 are given by (3) and (12) respectively. B_n is a functional of u so it has a Gateaux derivative B'_n. One proves (13) is a proper definition by induction from $n=3$ as follows. Note that $L_t=u_t=K_n$ and $L_t=[B_n, L]\Rightarrow K_n=[B_n, L]$, and similarly $C_n=[C_n, L]$. Then from (7), with σ_n for τ_n,

$$K_{n+1}=\frac{3}{n+1}(K'_n(\sigma_3)-\sigma'_3(K_n))$$

$$=\frac{3}{n+1}([B'_n(\sigma_3), L]+[B_n, L'(\sigma_3)]-[C'_3(K_n), L]-[C_3, L'(K_n)]). \tag{14}$$

Then from $L'(\sigma_3)=\sigma_3=[C_3, L]$, and similarly, and by using the Jacobi identity for $[\;,\;]$, one finds $K_{n+1}=B_{n+1}=[B_{n+1}, L]$ so that $u_t=K_{n+1}$ is $L_t=[B_{n+1}, L]$ for $n\geqslant 3$. One proves similarly $C_{m+1}=(3/(m-3))(C'_m(\sigma_3)-C'_3(\sigma_m)+[C_m, C_3])$ for $m\geqslant 4$ and $u_t=\sigma_n[u]$ is $L_t=[C_n, L]$ for $n=1,2,3,\ldots$.

To apply the non-local Riemann-Hilbert problem method to solve $u_t=K_{N+2}$, $u_t=\sigma_{N+2}$ we need some other results. The operators B_n, C_n are polynomials in D_x of degrees $n+1$, n. We assume that under the chosen boundary conditions $B_n\to\bar{B}_n$, $C_n\to\bar{C}_n$ as $x^2+y^2\to\infty$. Then from (13) and its analogue for $u_t=\sigma_n$ we find

$$\bar{B}_n=b_nD_x^{n+1} \qquad \bar{C}_n=yc_nD_x^n+xd_nD_x^{n-1}+e_nD_x^{n-2} \tag{15}$$

where $b_n=\frac{1}{3}(2i\sqrt{3})^n (n\geqslant 0)$, and c_n, d_n, e_n satisfy simple recurrence relations which yield

$$c_n=\tfrac{1}{3}(2i\sqrt{3})^{n-1} \qquad d_n=\tfrac{1}{3}(2i\sqrt{3})^{n-1} \qquad e_n=\tfrac{1}{6}(n-1)(2i\sqrt{3})^{n-2} \tag{16}$$

for $n\geqslant 1$, results we use shortly.

We turn now to the solution by the inverse spectral transform and the evolution of the spectral data.

KP-I is solved by the Riemann-Hilbert problem derived from $L\psi=0$ with L given by (11*a*) [4, 6]. Thus $u_t=K_n$, $n=0,1,2,\ldots$, is solved by the same method. Then, although $u_t=\sigma_{N+2}$, $N+2=1,2,\ldots$, is not isospectral, it is representable by $L_t=[L, C_n]$ so it is solved by the same method. Under the chosen boundary conditions KP-I has lump solutions [4, 6, 7] and there are both discrete and continuous spectral data. The continuous data $f(k, l, t)$ satisfy [4, 6]

$$\psi^+(k)=\psi^-(k)+\int_{-\infty}^{\infty}\psi^-(l)f(k, l; t)\,dl \tag{17}$$

where $\psi^\pm(k)$ solve $L\psi^\pm(k)=0$: they satisfy [4, 6] $\exp[-i(kx-\sqrt{3}k^2y)]\psi^\pm\equiv\mu^\pm\sim 1$ as $x^2+y^2\to\infty$ and $\mu^\pm$ can be analytically continued from the real k axis into the upper and lower half k planes respectively. The functions $\mu^\pm$ satisfy a Fredholm equation of the second kind with Green functions derived from $L\psi^\pm=0$, i.e. from $(D_x^2+k^2+D_y)G=-\delta(x)\delta(y)$ (refer to equations (7) (there given for $n=2$) and (5) of [6]). *Simple* poles in the eigenfunction $\mu^\pm$ at $k=k_j^\pm$ determine lump solutions [4, 6]. Then discrete data γ_j are defined through [4, 6]:

$$\lim_{k\to k_j^\pm}(\mu^\pm-i\psi_j^\pm(x, y)(k-k_j^\pm)^{-1})=[x-2\sqrt{3}k_j^\pm y+\gamma_j^\pm]\psi_j^\pm(x, y) \tag{18}$$

the residues $\psi_j^{\pm}$ are homogeneous solutions corresponding to $k_j^{\pm}$ of the Fredholm equation normalised by $(x-2\sqrt{3}k_j^{\pm})\ \psi_j^{\pm}\to 1$ as $(x^2+y^2)^{1/2}\to\infty$.

A complete set of spectral data S is [4-6] $S=\{f(k,l;t);\ k_j^{\pm},\ \gamma_j^{\pm},\ 1\leq j\leq M\}$. For its time evolution we first add a scalar multiple $\beta_n(k)$ of the identity to B_n: $\beta_n(k)$ is then essentially the linearised dispersion relation (as usual). Thus we set

$$L_n(k)\psi\equiv(D_t-B_n+\beta_n(k))\psi=0 \tag{19}$$

and since $\psi^{\pm}=\{\exp \mathrm{i}(kx-\sqrt{3}\,k^2y)\}\mu^{\pm}$, and $\mu^{\pm}\sim 1$, $B_n\sim\bar{B}_n$ as $x^2+y^2\to\infty$, $\beta_n(k)=(\mathrm{i}/6\sqrt{3})(-2\sqrt{3}k)^{n+1}$. Then, by applying $L_n(k)$ to (17) and using $L_n(k)-\beta_n(k)=L_n(l)-\beta_n(l)$ we readily find

$$f_t(k,l)=(\mathrm{i}/6\sqrt{3})((-2\sqrt{3}k)^{n+1}-(-2\sqrt{3}l)^{n+1})f(k,l) \tag{20}$$

while in a similar way we find

$$k_{j,t}^{\pm}=0 \qquad \gamma_{j,t}^{\pm}=\tfrac{1}{3}(n+1)(-2\sqrt{3}k_j^{\pm})^n \qquad j=1,\ldots,M. \tag{21}$$

Results (20) and (21) coincide for $n=2$ with those derived for KP-I [4, 6].

For the time evolution of the scattering data for $u_t=\sigma_{N+2}$ we define $L_n(k)$ through

$$L_n(k)\psi\equiv(D_t-C_n+\alpha_n(k))\psi=0. \tag{22}$$

From the asymptotics of $\psi^{\pm}$ and $C_n\sim\bar{C}_n$ as $x^2+y^2\to\infty$ we then find

$$k_t=(-1/6\sqrt{3})(-2\sqrt{3}k)^{n-1} \tag{23}$$

for all k in the k plane while $\alpha_n(k)=-\tfrac{1}{6}(n-1)(-2\sqrt{3}k)^{n-2}$. Consequently

$$\mathrm{d}f/\mathrm{d}t\equiv\tfrac{1}{2}(n-1)\tfrac{1}{3}[(-2\sqrt{3}k)^{n-2}-(2\sqrt{3}l)^{n-2}]f(k,l) \tag{24}$$

where $\mathrm{d}f/\mathrm{d}t\equiv f_t+f_kk_t+f_ll_t$ and k_t is given by (23): l_t is (23) with l replacing k.

For the discrete data we use (18). Equation (22) for $\psi^{\pm}$ is an equation for the $\mu^{\pm}$ and thus for the residues $\psi_j^{\pm}$ at their poles $k_j^{\pm}$. For large x, y (22) means $\bar{L}_n(k)\sim 0$ where

$$\begin{aligned}\bar{L}_n(k)=D_t-yc_n\sum_{m=0}^{n}\binom{n}{m}(\mathrm{i}k)^{n-m}\mathrm{D}_x^m-xd_n\sum_{m=0}^{n-1}\binom{n-1}{m}(\mathrm{i}k)^{n-m+1}\mathrm{D}_x^m\\ -e_n\sum_{m=1}^{n-2}\binom{n-2}{m}(\mathrm{i}k)^{n+m-2}\mathrm{D}_x^m\end{aligned} \tag{25}$$

with c_n, d_n, e_n from (16). Then, since from (18) $\lim_{k\to k_j^{\pm}}(k-k_j^{\pm})\mu^{\pm}=\psi_j^{\pm}$, application of $\bar{L}_n$ to this result followed by execution of the actual limit $k\to k_j^{\pm}$ yields

$$\bar{L}_n(k_j^{\pm})\psi_j^{\pm}=\lim_{k\to k_j^{\pm}}\frac{k_t-k_{j,t}^{\pm}}{k-k_j^{\pm}}\psi_j^{\pm}=\tfrac{1}{3}(n-1)(-2\sqrt{3}k_j^{\pm})^{n-2}\psi_j^{\pm} \tag{26}$$

after using (23) for k_t and $k_{j,t}$. Finally we apply $\bar{L}_n(k)$ to (18) itself; use (25) and (26), and take the limit $k\to k_j^{\pm}$ to find that (18) reduces to $\bar{L}_n(k_j^{\pm})(x-2\sqrt{3}\,k_j^{\pm}+\gamma_j^{\pm})$, so that, by using (26)

$$\gamma_{j,t}^{\pm}+\tfrac{1}{3}(n-1)(-2\sqrt{3}k_j^{\pm})^{n-2}\gamma_j^{\pm}=-\tfrac{1}{3}(n-1)(n-2)(-2\sqrt{3}\,k_j^{\pm})^{n-3}. \tag{27}$$

We thus have the time evolution of the spectral data S for $u_t=\sigma_{N+2}$, $n=N-2=1,2,\ldots$; if these equations can be solved the data S can be inverted (at fixed t) exactly as in the isospectral case.

By continuation so that $y \to -\mathrm{i}y$ KP-I becomes KP-II. For the latter it is therefore sufficient to replace y by $-\mathrm{i}y$ at each point of the analysis to (16). Then, although the method of solution becomes a $\bar\partial$ problem [5, 6] and data $T_1(k)$ [6] replace $f(k, l)$ and there are no discrete data (while μ and ψ are nowhere analytic and have no continuation), the present analysis is little changed. One finds

$$\mathrm{d}T_1/\mathrm{d}t = \tfrac{1}{2}(n-1)\tfrac{1}{3}[(-2\sqrt{3}\,k)^{n-2} - (2\sqrt{3}\,\bar{k})^{n-2}]T_1 \qquad n \geq 1 \tag{28}$$

where $\mathrm{d}T_1/\mathrm{d}t = T_{1,t} + T_{1,k}\,k_t + T_{1,\bar{k}}\bar{k}_t$; $k = k_\mathrm{R} + \mathrm{i}k_\mathrm{I}$, $\bar{k} = k_\mathrm{R} - \mathrm{i}k_\mathrm{I}$ and k_t, $\bar{k}_t$ are (23) in terms of k or $\bar{k}$.

We now consider the symplectic structure and constants of the motion, returning to KP-I and its hierarchies. The I_n of (8) are constants for $u_t = K_n$. The constants for $u_t = \sigma_n$ are constructed as follows [18]. We define

$$F_n(\lambda, t) \equiv \lambda^n(1 + \tfrac{1}{3}Nt\lambda^N)^{-(n+1)/N} = \sum_{m=0}^{\infty} f_{nm}\lambda^{n+Nm} \tag{29}$$

for $n = 0, 1, 2, \ldots$: $N \geq -1$ is a fixed integer to be identified with $n-2$; the f_{nm} depend on t; $\lambda \equiv (-2\sqrt{3}\,k)$. One checks

$$f_{n0} = 1; f_{nm+1,t} + \tfrac{1}{3}(n + Nm + 1)f_{nm} = 0 \tag{30}$$

and if $\bar{I}_n \equiv \Sigma_{m=0}^{\infty} f_{nm}I_{n+Nm}$, $n = 0, 1, 2, \ldots$, one finds

$$\frac{\mathrm{d}\bar{I}_m}{\mathrm{d}t} = \frac{\partial}{\partial t}\bar{I}_m + \{\bar{I}_m, J_{N+2}\} = 0 \tag{31a}$$

while

$$\{\bar{I}_m, \bar{I}_n\} = 0 \qquad m, n = 0, 1, 2, \ldots. \tag{31b}$$

Thus the $\bar{I}_m$ are constants with respect to the Hamiltonian J_{n+2} and the flow $u_t = \mathrm{D}_x\delta J_{N+2}/\delta u = \sigma_{N+2}$. They are in involution, and since $N \geq -1$ is an arbitrary integer there are distinct countably infinite sets of $\bar{I}_m$ for each N and each flow $u_t = \sigma_{N+2}$. This does not (quite) prove $u_t = \sigma_{N+2}$ is completely integrable. But the I_n can be expressed in terms of scattering data [7] and for each N the $\bar{I}_n$ therefore can be. We then find the $\bar{I}_n \equiv \bar{I}_n^{(N)}$ can be expressed in terms of action-type variables alone with a determinable time evolution. We give the details elsewhere (some details are in [18]) and show there also that there are the further quantities $\bar{J}_m^{(N)}$ and symmetries $\bar{K}_m^{(N)}$, $\bar{\tau}_m^{(N)}$ of $u_t = \sigma_{N+2}$ satisfying the Lie algebra (10) mentioned earlier in this letter. We should note that $N = n-2$ only becasue we work from KP-I with symmetry $K \equiv K_2$: from $K \equiv K_m$ we derive flows $u_t = \tau_{(n-m)+m}$ and $N = n - m$. However, the $\bar{J}_m^{(N)}$ are linear combinations of the I_n and satisfy a relation like that for the $\bar{I}_m^{(N)}$. Thus there is no point in pursuing flows and symmetries $u_t = \bar{K}_m^{(N)}$, $u_t = \bar{\tau}_m^{(N)}$ for each N as (10) suggests. There are thus the flows $u_t = K_n$ and $u_t = \sigma_n$, and only these, of independent significance. We complete this letter with some examples.

A general set of integrable NEE is evidently

$$u_t = \sum_{n=0}^{M} \omega_n K_n + \sum_{n=1}^{M'} \bar{\omega}_n\sigma_n \tag{32}$$

with Lax operator L given by (11a) and B given by $B = \Sigma_{n=0}^{M}\omega_n B_n + \Sigma_{n=1}^{M'}\bar{\omega}_n C_n$. These NEE are solved by the spectral method sketched earlier in this letter and the spectrum evolves as $\lambda_t = \frac{1}{3}\Omega(\lambda)$; $\Omega = \Sigma_{n=1}^{M'}\bar{\omega}_n\lambda^{n-1}$ with $\lambda \equiv -2\sqrt{3}\,k$. More particularly consider

$$\begin{aligned} u_t &= K_2 + \alpha\sigma_1 + \beta\sigma_2 \\ &= \mathrm{D}_x^{-1}u_{yy} - 6uu_x - u_{xxx} + \tfrac{1}{3}\alpha y u_x + \beta(\tfrac{1}{3}xu_x + \tfrac{2}{3}yu_y + \tfrac{2}{3}u) \end{aligned} \tag{33}$$

in which α, β are constants. The Lax pair operator is $B = B_2 + \alpha C_1 + \beta C_2$, and the time evolution of the spectral data is

$$\begin{aligned}
&(-2\sqrt{3}\,k_j^{\pm})_t = \tfrac{1}{3}\alpha + \tfrac{1}{3}\beta(-2\sqrt{3}\,k_j^{\pm}) \\
&\gamma_{j,t}^{\pm} = 12k_j^{\pm} - \tfrac{1}{3}\beta\gamma_j^{\pm} \\
&f_t = 4\mathrm{i}(l^3 - k^3)f + (6\sqrt{3})^{-1}\alpha(f_k + f_l) - \tfrac{1}{3}\beta(kf_k + lf_l).
\end{aligned} \tag{34}$$

Moreover the Hamiltonian is $H = I_2 + \alpha J_1 + \beta J_2$ (determined by (8) and (9)) and the constants of the motion are

$$\bar{I}_n = \exp[-\tfrac{1}{3}\beta(n+1)t] \sum_{m=1}^{n+1} \binom{n+1}{m} \alpha^{n+1-m} \beta^m I_{m-1}. \tag{35}$$

Further details of the results given in this letter as well as some of their generalisations mentioned will be given elsewhere.

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Note added in proof. In connection with the Virasoro type algebras of (7) and (10), it has been shown [22] that symmetries like the τ_m in 1+1 dimensions can be realised as infinitesimal conformal transformations of the spectral problem: no equivalent result for the KP equations (which are also discussed) was reported.

References

[1] Kadomtsev V V and Petviashvili V I 1970 *Sov. Phys.-Dokl.* **15** 539
[2] Bullough R K and Caudrey P J (ed) 1980 *Solitons* (*Topics in Current Physics* **17**) (Berlin: Springer)
[3] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons: the Inverse Scattering Method* (New York: Plenum)
[4] Fokas A S and Ablowitz M J 1983 *Stud. Appl. Math.* **69** 211
[5] Ablowitz M J, Bar Yacov D and Fokas A S 1983 *Stud. Appl. Math.* **69** 135
[6] Jiang Z and Bullough R K 1987 *J. Phys. A: Math. Gen.* **20** L429
[7] Jiang Z, Bullough R K and Manakov S V 1986 *Physica* **18D** 305; 1988 to be published
[8] Zakharov V E and Schulman E I 1980 *Physica* **1D** 192
[9] Oevel W and Fuchssteiner B 1982 *Phys. Lett.* **88A** 323
[10] Chen H H, Lee Y C and Lin J E 1983 *Physica* **9D** 439
[11] Goddard P and Olive D 1986 *Int. J. Mod. Phys.* A **1** 303
[12] Case M 1985 *J. Math. Phys.* **26** 1158
[13] Li Yi-shen and Zhu Guo-cheng 1986 *J. Phys. A: Math. Gen.* **19** 3713
[14] Calogero F and Degasperis A 1987 *Lett. Nuovo Cimento* **22** 138
[15] Lakshmanan M and Bullough R K 1980 *Phys. Lett.* **80A** 287
[16] Lakshmanan M and Ganesan S 1985 *Physica* **132A** 117
[17] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943
Date E, Kashiwara M and Miwa T 1981 *Proc. Japan. Acad.* A **57** 387
[18] Yi Cheng and Yi-shen Li 1987 *Physica* D in press
[19] Zakharov V E and Konopelchenko B G 1984 *Commun. Math. Phys.* **94** 483
[20] Chen H C, Lee V C and Lin S C 1979 *Phys. Scr.* **20** 490
[21] Santini P M and Fokas A S 1986 *Preprints* Recursion Operators and Bi-Hamiltonian Structures in Multidimensions I, II
[22] Orlov A Yu and Schulman E I 1986 *Lett. Math. Phys.* **12** 171